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NAVAL POSTGRADUATE SCHOOL

Monterey, California

AN ALLOCATION MODEL

FOR ATTACKING DEFENDED TARGET COMPLEXES

WITH IMPERFECT ATTACKERS

by

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ABSTRACT

A model is presented for the attack of two defended target complexes with a fixed force of imperfect missiles. The attackers will be directed first at the defensive system then at the targets themselves. The imperfect defenders are used against the attackers on a one-for-one basis as long as defenders remain. If any attacker penetrates the defensive system, all the defenders at that target complex are destroyed. The problem addressed is the offensive problem of determining how many attackers to send to each defensive system and to each target complex. The necessary mathematical relationships are derived and used to solve a sample problem.

Prepared by:

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Executive Summary.

The purpose of the research reported here is to analyze a ballistic missile attack against defended targets with the purpose of determining effective targeting tactics for the offense.

Basically we consider two target complexes each containing a known number of targets. Each complex is defended by a known number of defensive missiles each of which can defend any target in the complex.

The attacking missiles can be directed at the defensive missiles or at the target themselves. If an attacking missile is aimed at the defensive system and it penetrates the defense, it destroys the entire defensive complex. The attack is assumed to be sequential. The offense first commits some number of its attackers to the defensive system, then it attacks the targets. The defense is assumed to be one-on-one, but this can be modified. Both the offensive and defensive missiles are imperfect, each working with some probability.

The offensive problem is to determine how its fixed force of attackers should be allocated between the two target complexes and how many of the attackers should be allocated to the defensive system at each complex. We assume that the offense receives no information about the success or failure of its weapons in the course of the attack. The measure of effectiveness used is to maximize the expected number of targets destroyed.

This report describes the computations involved in the allocation model and solves a sample problem for illustration. Several generalizations are mentioned.

Measure of Effectiveness.

This section discusses the choice of the measure of effectiveness used in the allocation model. The ballistic missile attack is analyzed from the offensive point of view. The objective of the analysis is to determine tactics which will permit the offense to use his forces more effectively. The term "more effective use of resources" must be translated into terms which can be used unambigously to guide the offense in its weapon deployment.

Aside from the deterrent effect, the purpose of the offensive system is to destroy targets. It would be desirable in an actual attack to destroy, if possible, the most valuable set of targets, but then we have the problem of determining or assigning target values. No general agreement can be reached regarding the values to be assigned; and even if agreement could be reached, any values assigned could not reflect interactions between targets. For example, the value of an industrial target depends very much on the continued existence of a power plant to run it.

It is assumed that if some targets have a value which is obviously large compared to most of the others, these targets will be given special consideration in targeting. The majority of the targets, however, are assumed to be of roughly comparable value and the criterion used in this report is to maximize the expected number of targets destroyed. It is assumed here that the targets do not vary in value with time, but that case is discussed briefly in section 6.

For planning purposes on a larger scale it is possible that a more versatile measure of effectiveness would be desired, but for the examination of alternative tactics the criterion of maximizing the expected number of targets destroyed serves as a useful means of comparing alternatives.

Problem Description.

The basic problem addressed in this report is the problem of allocating a fixed force of imperfect offensive missiles to a fixed set of targets. There are two groups of targets or target complexes. Within each complex is some fixed number of targets known to the offense. Each complex is defended by known numbers of imperfect defensive missiles each of which can be used against any missile approaching any target in that complex.

The attacking missiles can be directed to either the targets themselves or to the defensive missile launching complex, probably the control radars. It is assumed that if an offensive missile which is aimed at the defensive complex penetrates the defense and hits its target, the entire force of defensive missiles is rendered useless. If an offensive missile is destroyed or misses its intended target it does no damage at all.

The offensive problem is to determine how many missiles to direct toward each of the two target complexes and how many of these should be allocated to the defensive missiles and how many to the actual targets.

The attack can be thought of as sequential, the offense first directing some number of attackers to the defensive system and then the remainder to the targets themselves. We assume that the offense has no damage assessment capability; that is, he can not tell which missiles if any, have successfully penetrated to their targets. We also assume in the report that the defense does not have the capability of attack evaluation; that is, he can not determine in flight the impact point of an incoming missile accurately enough that he dares to let it pass undefended with the knowledge that it will impact harmlessly. Even if the defense can determine the impact point he is assumed here to be unable to correlate that information in real time with the continued existence or previous death of targets near the impact point. Thus we assume that as long as the defense has missiles available he will not let offensive missiles proceed undefended. We assume that the defense is one-on-one, but this can easily be modified to include other possibilities.

The model is an offense-last-move model and assumes that the offense knows both the number of targets and the number of defensive missiles in each complex. Thus the offense will never allocate more attackers to the defensive system than the number of defensive missiles because the supply of defensive missiles will be exhausted at that point anyway.

Since the offense has no damage assessment capability the best procedure for him to follow is to spread as evenly as possible over the targets those re-entry vehicles which are allocated to targets.

The basic problem then is to determine the optimal allocation of the fixed force of offenders to the defensive systems and the target complexes to maximize the expected number of targets destroyed. See Figure 1.

For a review of the literature on missile allocation problems see [2] and [3].

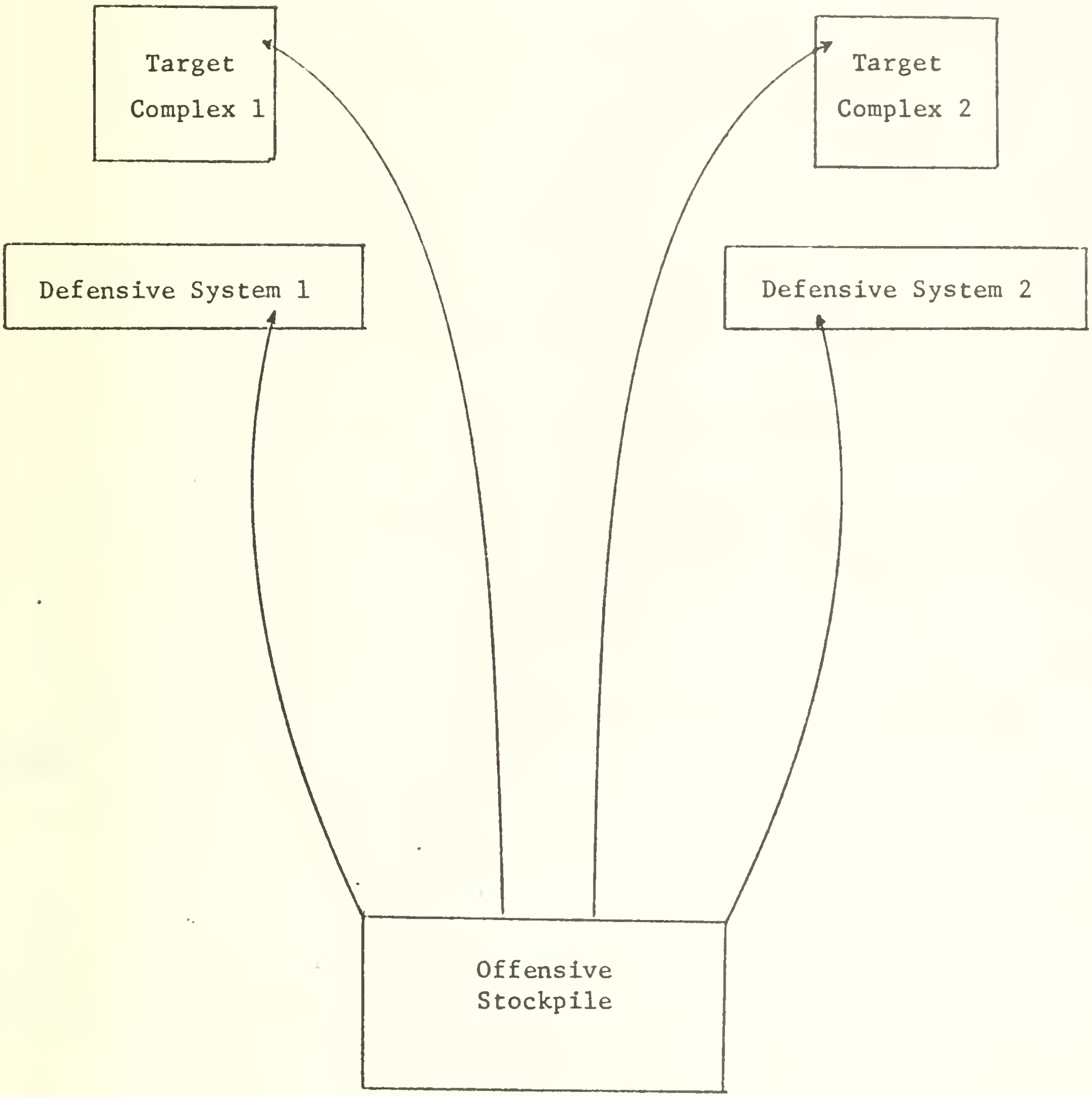

Figure 1.

The Allocation Model

Analysis.

We will deal first with a single target complex and develop the necessary relationships and then we will examine the allocation problem between complexes. We let

t = the number of targets in the complex,

t' = the number of defenders guarding the complex,

A = the total number of attackers,

a = the number of attackers assigned to targets,

a' = the number of attackers assigned to defenders,

p_a = probability that an attacking missile kills its target when no defender is used,

p_d = probability that a defensive missile which is assigned to an attacker kills that attacker,

$p_k = p_a(1-p_d)$, the probability that an attacking missile kills its target when a defender is used.

Case 1: $a \leq t$

We consider first the case in which these are sufficiently many targets so that regardless of how the attackers are split between defenders and targets every attacker assigned to a target is assigned to a live target.

If any of the first a' attackers penetrates the defense and kills its target, it destroys all the defenders in which case the number remaining is zero. On the other hand, if none of the first a' attackers penetrates, the number of defenders remaining is exactly $t' - a'$.

Let p_i = probability that exactly i defenders remain available for use after the first a' attackers complete the attack on the defensive system.

We have that

$$p_i = 0 \qquad i \neq t' - a'$$

$$p_i = (1-p_k)^{t'-i} \qquad i = t' - a'.$$

Number the attackers beginning with the first one which is assigned to a target and let

p_j' = probability that attacker j kills a target.

We have

$$p_j' = p_a \cdot \{\text{probability the target is defended}\}$$
$$+ p_k \cdot \{\text{probability the target is not defended}\}.$$

The probability that the $j^{\underline{th}}$ target is defended is the probability that j or more defenders remain available for use after the initial attack on the defensive system. Call this probability P_j.

We have

$$P_j = \sum_{i=j}^{t'} p_i$$

so that

$$p_j' = p_a(1-P_j) + p_k \cdot P_j$$

which can be rewritten as

$$p_j' = p_a\left[1 - \begin{cases} (1-p_k)^{a'}, & j \le t' - a' \\ 0 \quad , & j > t' - a' \end{cases}\right]$$
$$+ p_k\left[\begin{cases} (1-p_k)^{a'} \quad , & j \le t' - a' \\ 0 \quad , & j > t' - a' \end{cases}\right]$$

or as

$$p_j' = p_a + (p_k - p_a)\cdot\begin{cases}(1-p_k)^{a'}, & j \le t' - a' \\ 0 \quad , & j \ge t' - a'\end{cases} \quad .$$

The expected total number of targets killed in the complex will be written as $E(t',A,a')$ and is

$$E(t',A,a') = \sum_{j=1}^{A-a'} p_j' \; .$$

Using the expression for p_j' above gives

$$E(t',A,a') = (A-a')p_a + (p_k - p_a)(t'-a')(1-p_k)^{a'} .$$

Analytical efforts to maximize $E(t',A,a')$ over a' have not been successful, but for fixed values of p_a, p_d, A, and t' it is easy to compute $E(t',A,a')$ for all $a' \le t'$. The results can be plotted and the best value of a' determined. This has been done for a few sample cases and the results are shown in Figures 2-5.

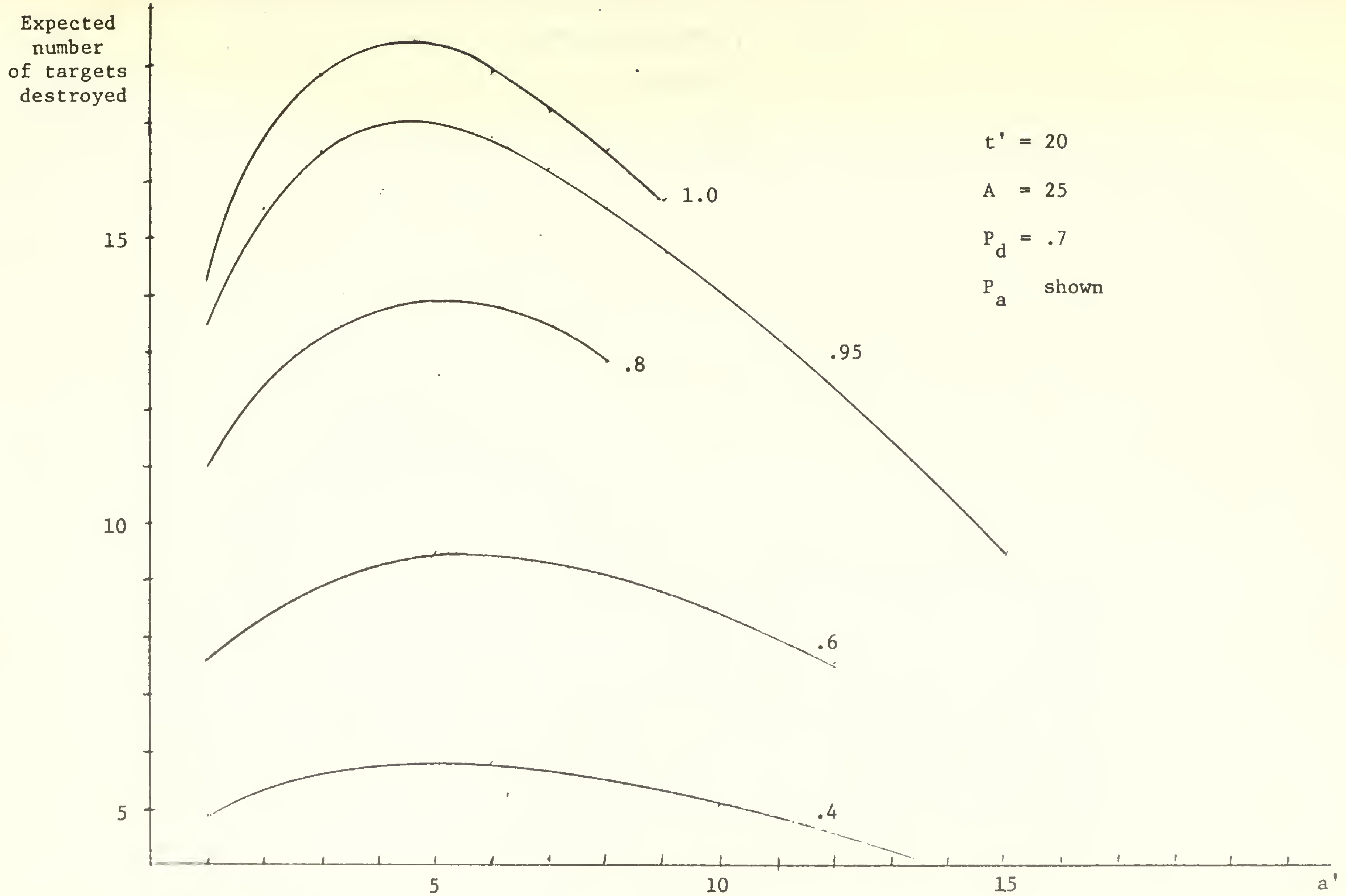

Figure 2.

Results for Case 1, $a \le t$.

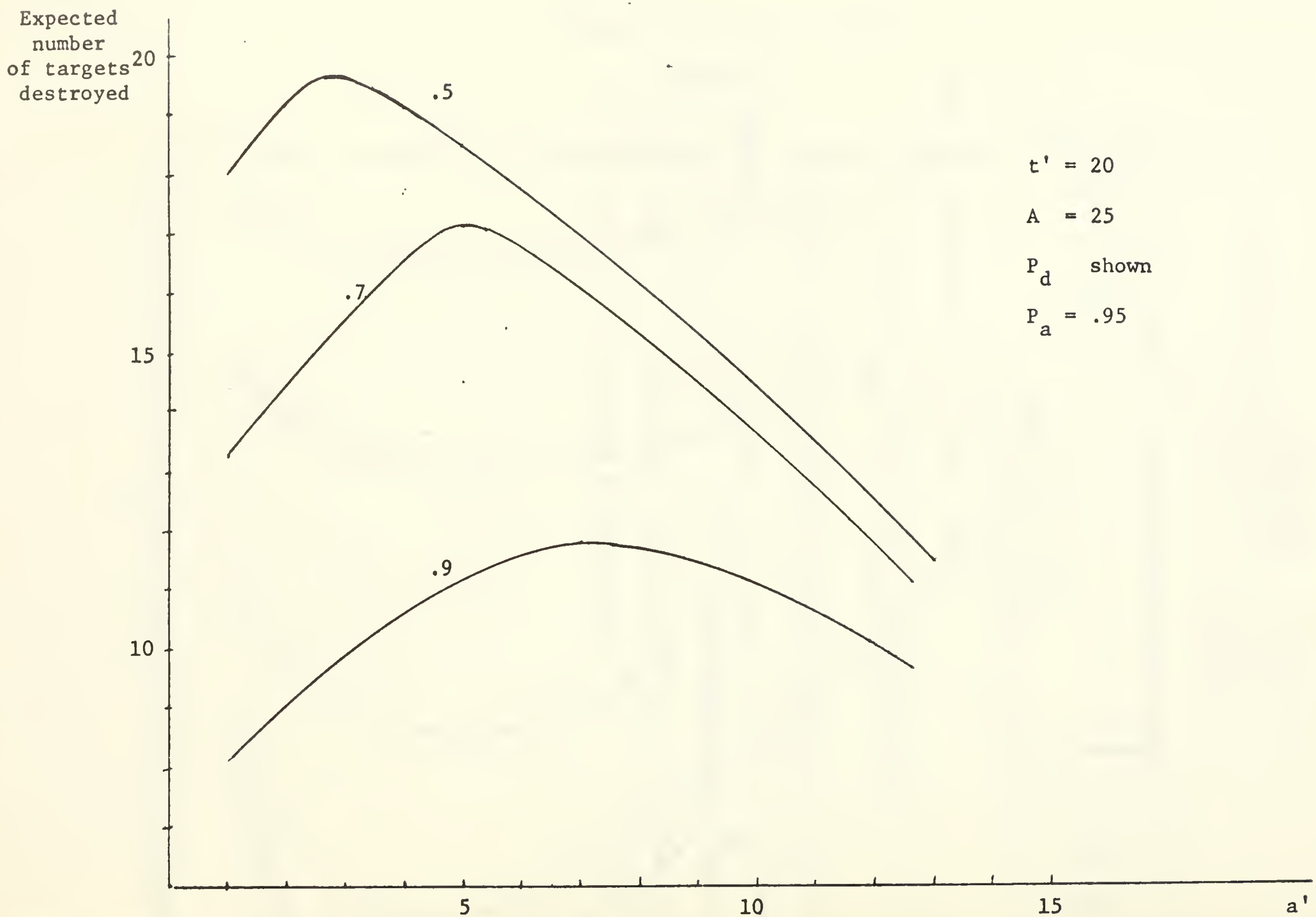

Figure 3.

Results for Case 1, $a \leq t$.

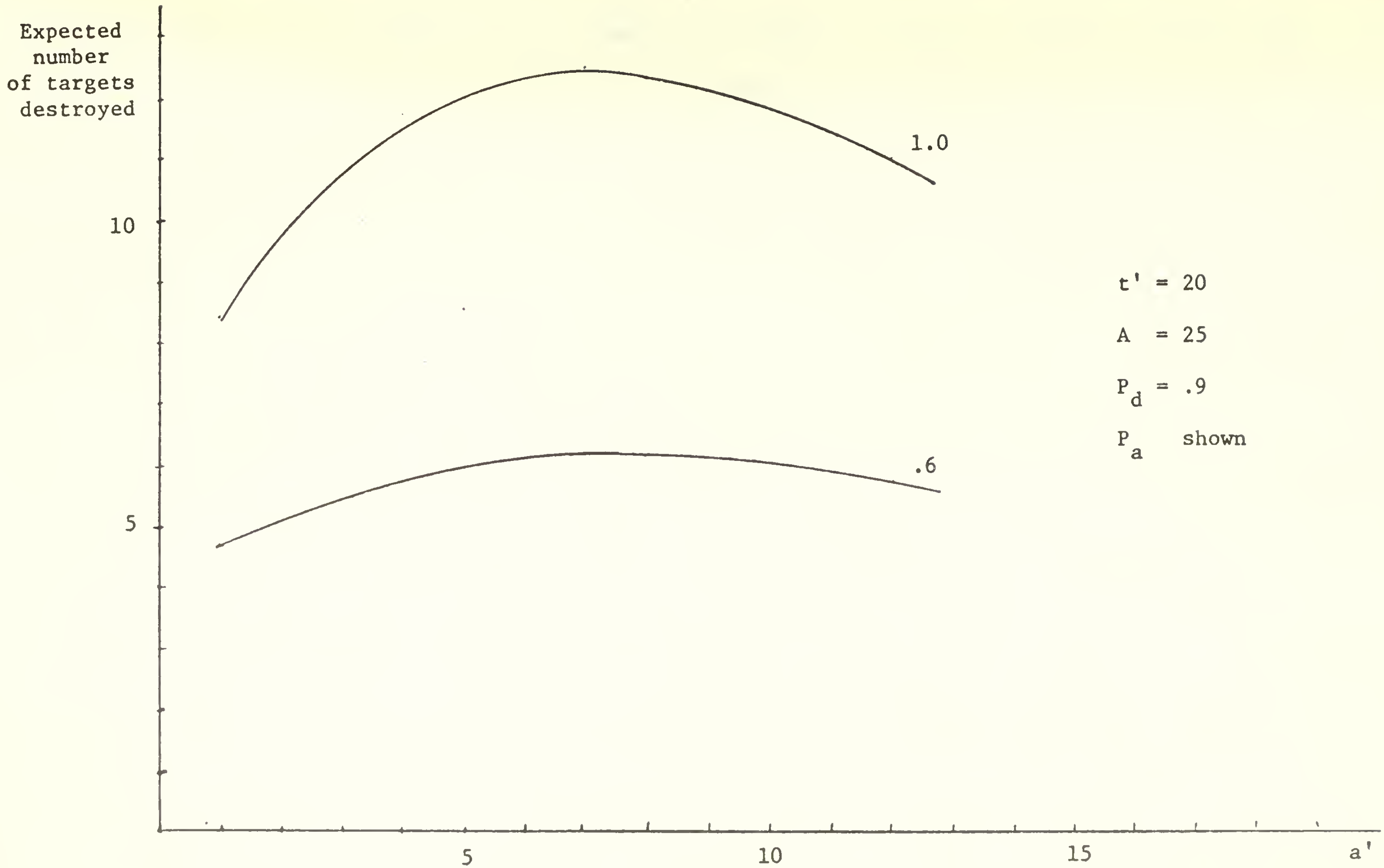

Figure 4.

Results for Case 1, $a \leq t$.

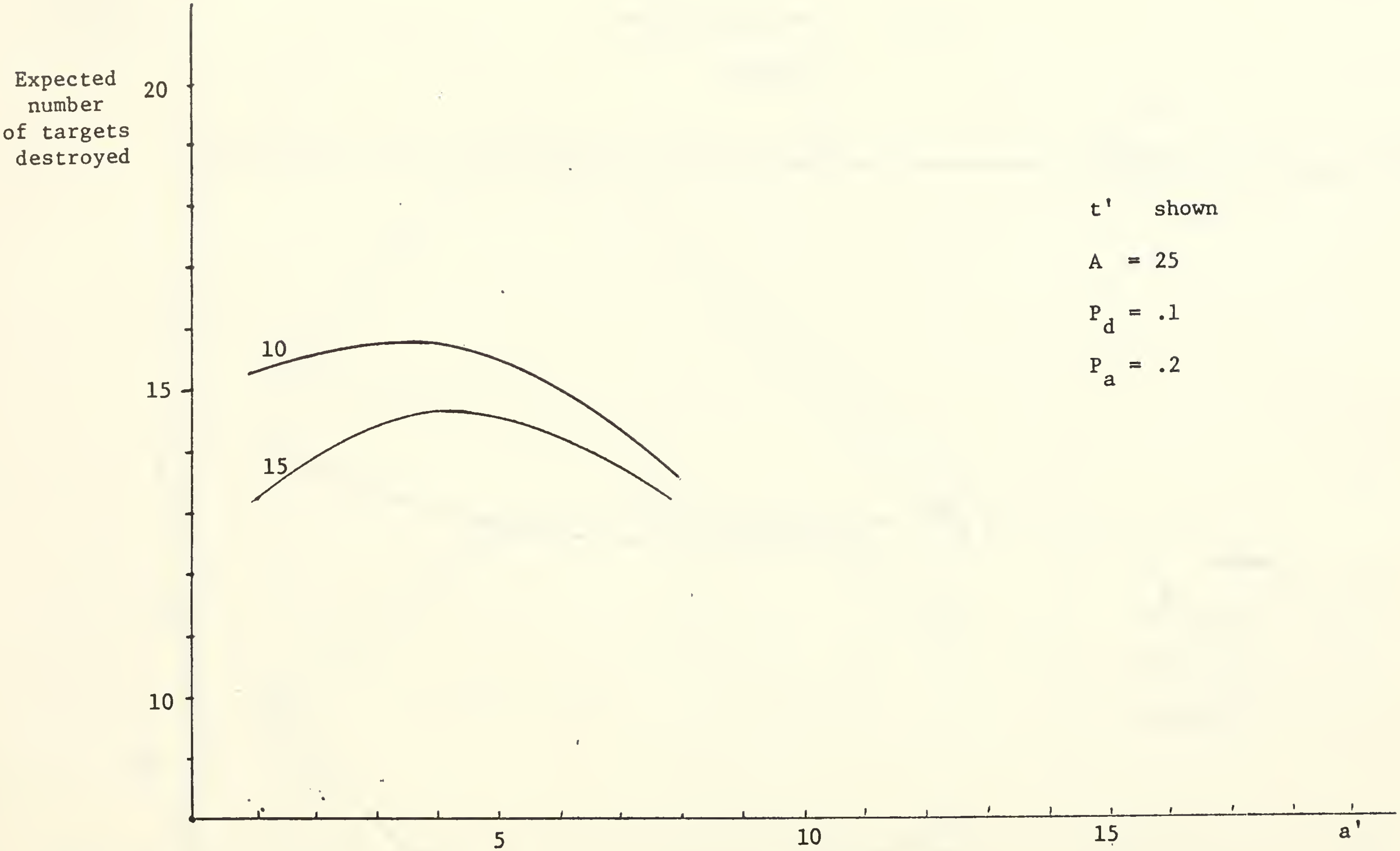

Figure 5.

Results for Case 1, $a \leq t$.

Case 2: $a > t$

In this case we consider the fact that if the number of attackers substantially exceeds the number of targets, many attackers will be aimed at targets previously killed and the computations in case 1 will give an inflated estimate of the number of targets killed.

The modification required here is that the probability of kill for each attacker must be weighted by the probability that the target is alive. We have

$$p_j' = \begin{bmatrix}\text{Probability no defenders survive the} \\ \text{initial attack on the defensive system}\end{bmatrix} \cdot p_a \cdot \left[\bar{p}_j \middle| \begin{matrix}\text{no defenders} \\ \text{survive}\end{matrix}\right]$$

$$+ \begin{bmatrix}\text{Probability (t'-a') defenders survive} \\ \text{initial attack on the defensive system}\end{bmatrix} \cdot p_{ak} \cdot \left[\bar{p}_j \middle| \text{defenders survive}\right]$$

where, as before

p_j' = probability that the j^{th} attacker kills a target,

and where

p_{ak} = probability that the j^{th} offensive weapon imparts near enough to the target to kill it. $p_{ak} = p_a$ if the j^{th} attacker is undefended and $p_{ak} = p_k$ if the j^{th} attacker is defended,

and

$\bar{p}_j$ = probability that the intended target is alive before the arrival of the j^{th} attacker.

It is appropriate to review the assumptions about how the attack proceeds. We assumed that the offense shoots sequentially a' missiles at the defensive system. The defense fires one defender at each of these attackers providing he is capable of doing so. After the initial a'

attackers are sent, the offense begins to fire sequentially at the t targets. Since he receives no information about any target destroyed he is unable to modify his attack plans so he simply fires at the targets in order repeating the attack as long as his resources allow. Thus he spreads his attackers as evenly as possible over the targets. If any defenders survive the initial attack, the defense continues to fire one defender at each of these attackers as long as his resources allow.

Letting $E(t,t',A,a')$ be the expected number of targets killed in case two, we have

$$E(t,t',A,a') = \sum_{j=1}^{A-a'} p_j'$$

$$= \{1-(1-p_k)^{a'}\} \cdot \{ \sum_{j=1}^{A-a'} p_a(1-p_a)^{[(j-1)/t]} \}$$

$$+ \{(1-p_k)^{a'}\} \cdot \{ \sum_{j=1}^{t'-a'} p_k(1-p_k)^{[(j-1)/t]} + \sum_{j=t'-a'+1}^{A-a'} p_a(1-p_k)^{n_j}(1-p_a)^{m_j} \}$$

where

$[x]$ = largest integer in x,

n_j = number of times the target of attacker j has been previously attacked and defended, $j > t' - a'$, given that defenders remained after the initial a' attackers,

m_j = number of times the target of attacker j has previously been attacked and not defended, $j > t' - a'$, given that defenders remained after the initial a' attackers.

The expressions for n_j and m_j are derived below. They apply to the case where the number of defenders available after the initial a' attackers is $t' - a'$ which is the number remaining when the attackers have failed to destroy the defensive complex.

Let

$$k_1 = [(t'-a')/t], \qquad \text{(largest integer)}$$

and

$$r_1 = (t'-a') - k_1 t.$$

Thus $t - r_1$ targets have exactly k_1 defenders and r_1 targets exactly $k_1 + 1$ defenders. We get for $j > t' - a'$

$$n_j = k_1 + \begin{cases} 1, & j - [(j-1)/t] \leq r \\ 0, & \text{otherwise.} \end{cases}$$

The total number of attackers previously (before the $j^{\underline{th}}$) fired at the target of the $j^{\underline{th}}$ attacker is $[(j-1)/t]$ and defenders were used against n_j of these so the number against which there were no defenders is

$$m_j = [(j-1)/t] - n_j, \qquad j > t' - a'.$$

As in case 1 efforts to determine the maximum of $E(t,t',A,a')$ over a' have not proved successful but a program was written to compute E for fixed values of t, t', A, p_a, p_d and for all values of a' from 1 to t'. The results of some sample computations are shown in Figures 6 and 7.

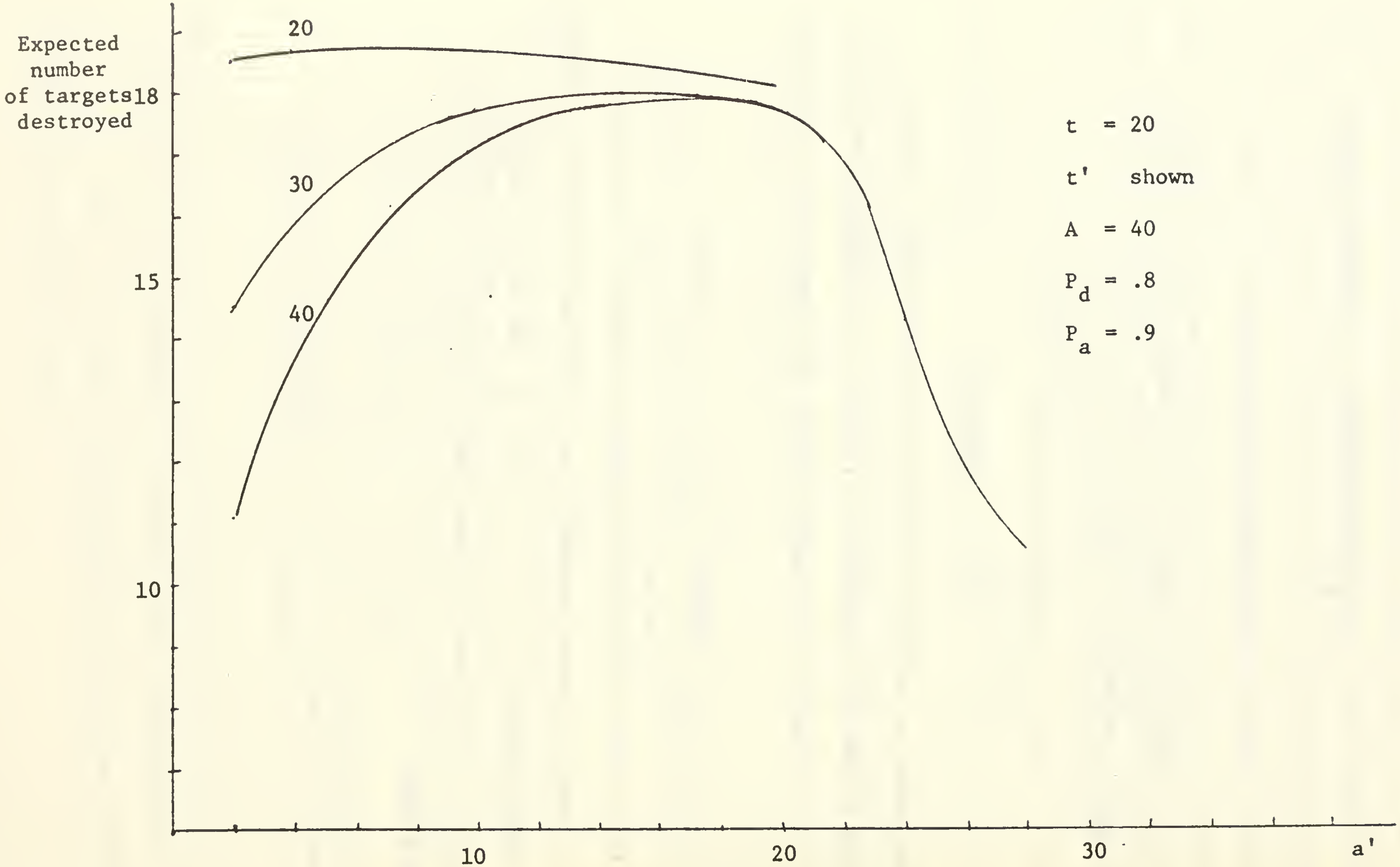

Figure 6.

Results for Case 2, $a > t$.

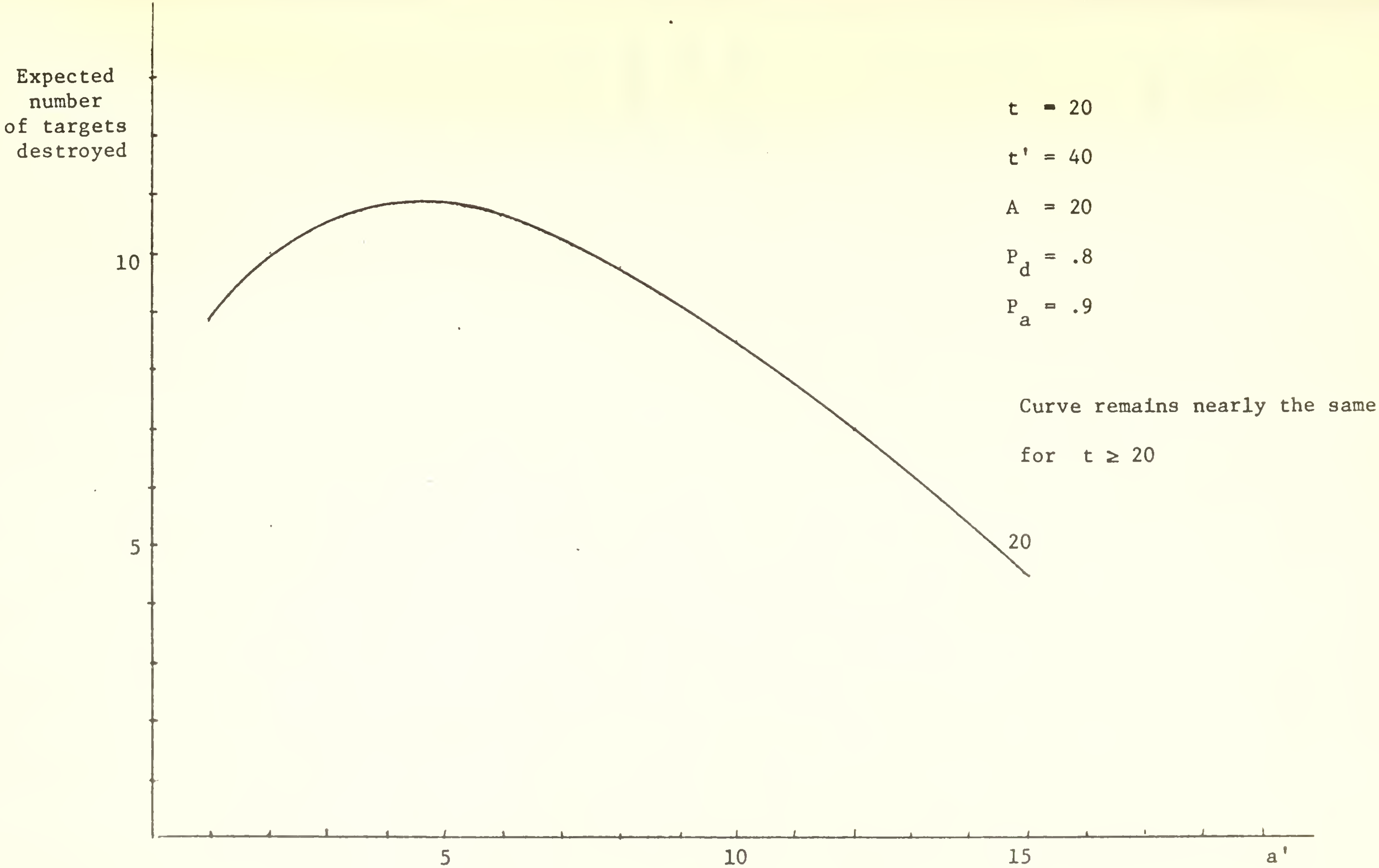

Figure 7.

Results for Case 2, $a > t$.

Allocation Model.

Using the results of the previous sections (case 2) a model was constructed to evaluate how a fixed number of attackers should be allocated to two target areas each containing a known number of targets and defended by a known number of defenders. The offensive decision problem is pictured in Figure 8.

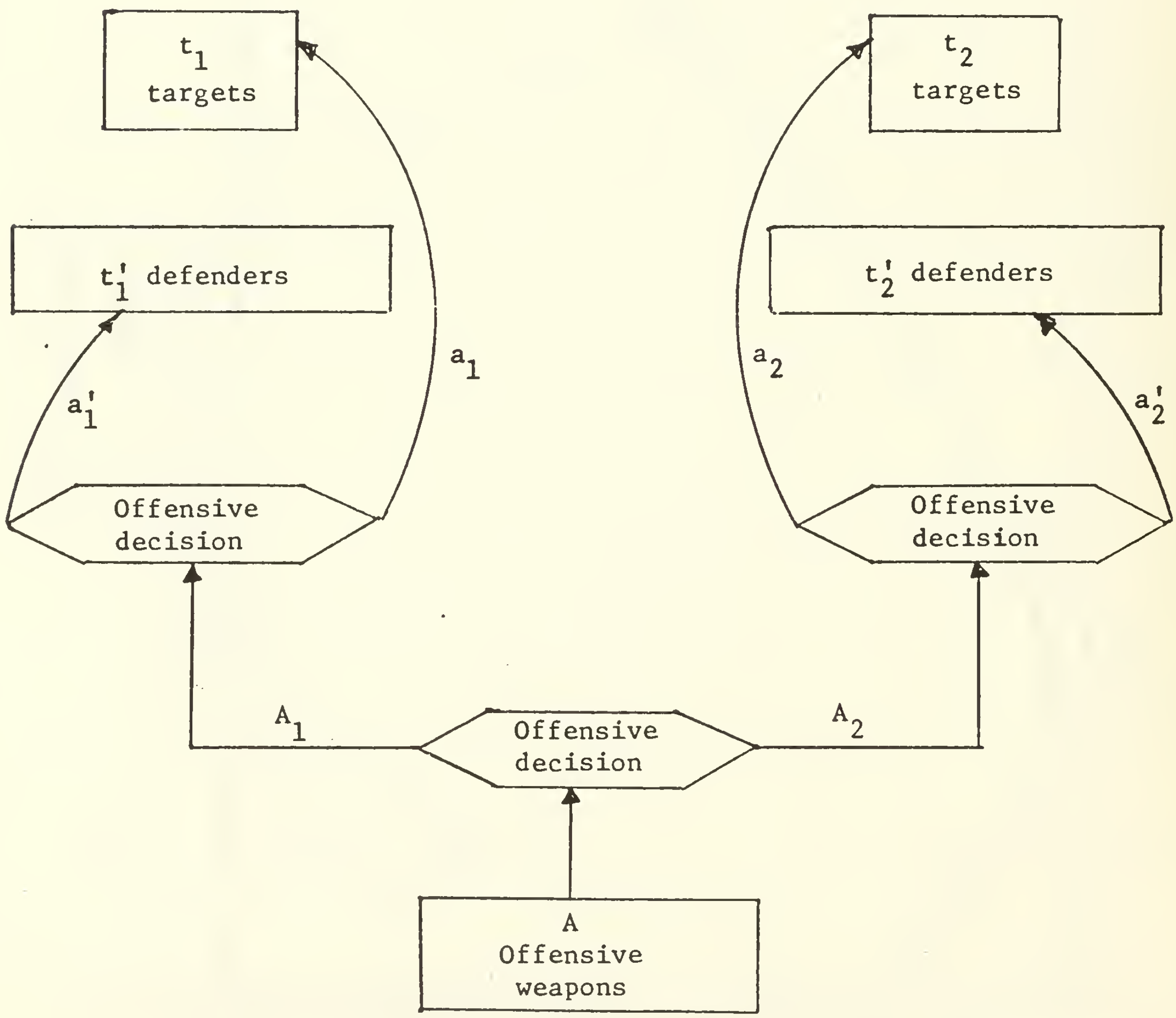

Figure 8.

Offensive decision problem.

We have $E(t_i,t_i',A_i,a_i')$, $i = 1,2$ as the expected number of targets killed in area i given t_i targets, t_i' defenders, A_i attackers a_i' of which are allocated to the defensive system.

Let

$$E^*(t_i,t_i',A_i) = \max_{a_i' \le t_i'} E\,(t_i,t_i',A_i,a_i') \qquad i = 1,2.$$

The maximum expected total number of targets killed in the two target complexes when A_1 are allocated to complex 1 and A_2 are allocated to complex 2 is

$$f(A_1,A_2) = \sum_{i=1}^{2} E^*(t_i,t_i',A_i).$$

The allocation problem is to maximize $f(A_1,A_2)$ subject to $A_1 + A_2 = A$. This can easily be accomplished numerically for particular values of the parameters. The results of a sample computation are given in Table 1. The sample problem solved for illustration is pictured in Figure 9.

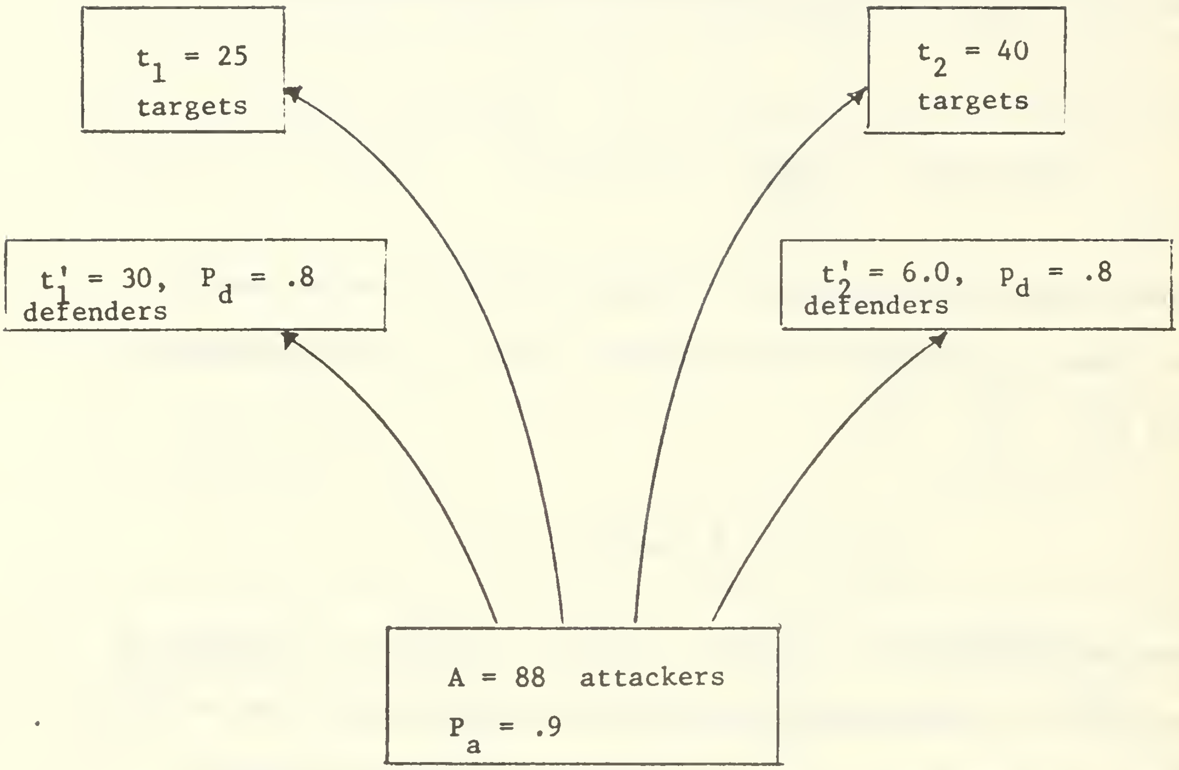

Figure 9.

Sample Problem.

The assumption was made in doing the computations that $f(A_1,A-A_1)$ is a unimodal function of A. This assumption has not been verified, but numerical work supports it. The assumption is equivalent to assuming that the farther A_1 is from the optimal value the worse the system performance is, but the assumption makes no other restriction on the shape of the function nor any on the location of the maximum. With this assumption a Fibonacci search procedure can be used to search for the maximum of $f(A_1,A-A_1)$, see [4].

The computations were done for each value of A_1 selected by adding the maximum number of targets destroyed in complex 1 to the maximum number destroyed in complex 2. These quantities were computed using the model described in case 2. Their sum gives $f(A_1,A-A_1)$ for the particular value of A_1.

The values of A_1 considered in the Fibonacci search are shown in Table 1 in the order required along with the supporting computations.

A_1	$A-A_1$	$E^*(t_1,t_1',A_1)$	a_1'	a_1	$E^*(t_2,t_2',A_2)$	a_2'	a_2	$f(A_1,A-A_1)$
34	54	19.96	9	25	34.61	15	39	54.57
55	33	24.16	7	48	20.70	7	41	44.86
21	67	11.00	5	16	36.27	20	47	47.27
42	46	22.19	16	26	30.06	9	37	52.25
29	59	16.12	7	22	35.43	20	39	51.58
37	51	21.30	12	25	33.51	12	39	54.81
39	49	21.78	14	25	32.34	10	39	54.12
36	52	20.96	11	25	33.95	13	39	54.91
35	53	20.52	10	25	34.31	13	40	54.83

Table 1.

Sample Computations for the Allocation Model.

These results are pictured in Figure 10 along with the value of $f(A_1,A-A_1)$ for $A_1 = 76$.

It is interesting to compare these results with the results of the purely subtractive model whose solution would be to fire all 88 weapons at target complex 2 and would result in the destruction of 28 targets.

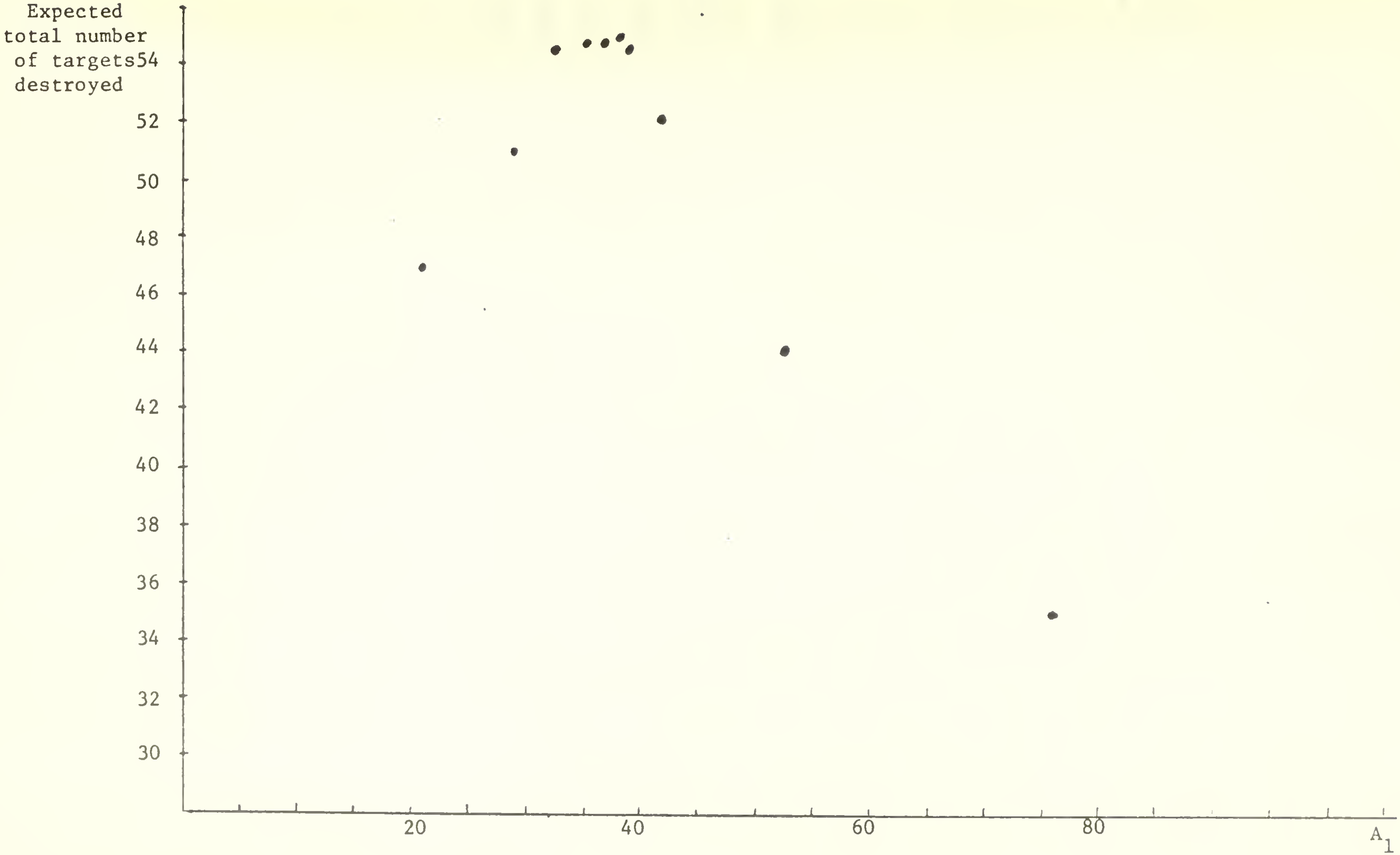

Figure 10.

Results from sample problem.

Discussion.

The objective function used in this report deals only with the expected total number of targets destroyed and is subject to the criticism that it does not deal with the distribution of the actual number of targets destroyed. That is, after the attack on the defensive complex is completed, either the defenders have been destroyed or they have not and the subsequent battle will differ considerably in these two cases. The decision maker's risk aversion can be dealt with, to some extent, by providing him with additional information about the expected number of targets destroyed. Instead of providing only the expected number, we can also provide a curve which shows as a function of a' the expected number of targets destroyed for each of these two conditions.

The objection that the actual number of targets destroyed may differ considerably from the expected number and that using the expected number may lead to unacceptable decisions is a reflection of the fact that the decision maker has a marginal utility which is decreasing with the number of targets destroyed. Such a utility function could be included in this decision making problem.

Generalizations.

A number of interesting generalizations are possible in this model. One easy extension is to permit p_a (and p_d) to differ for the two target complexes. This difference might arise because the targets would be attacked from different launch areas.

Another easy generalization would be to consider variations in the defensive interceptor commitment policy. For example, we could consider two-on-one defense for all attackers directed at the defensive complex.

Slightly more difficult generalizations involve changing the structure of the defensive complex to consist of several launcher groups. Similarly, we could assume that the defenders are not rendered useless until two attackers penetrate. This would be relevant for the case where each launcher group has two control radars, either of which can control the interceptors.

Generalizations could also be made by changing the assumption about the knowledge available to the offense and defense. If the defense has attack evaluation capability, his performance will be improved; or if the offense has damage assessment capability he can increase the expected number of targets destroyed.

A very interesting and apparently difficult extension is to assume that some of the targets have a value which diminishes with time. If some of the targets are offensive missile launchers there is no benefit in attacking the launcher after the missile is gone. A model dealing with decreasing target values is presented in [1].

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Ballistic missile attack						
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Targeting model						